ON V. ZEIPEL'S THEORY OF THE PERTURBATIONS OF MINOR PLANETS OF THE *HECUBA* GROUP.

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By reason of the difficulties encountered in determining the perturbations of certain minor planets, and in recognition of a practical analytical treatment by groups, we adopt a natural and useful classification according to their common characteristic. This common characteristic is the near commensurability of the planets' mean motion with that of the primary disturbing body, *Jupiter.*

Let *Jupiter's* mean daily motion be n', that of the planet n. Whenever the ratio

$$\mu = \frac{n'}{n}$$

is approximately the ratio of two low integers, practical difficulties arise.

Now *Jupiter's* mean motion is 299″, and the planets of the group to which *Hecuba* belongs have mean motions near 600″. The ratio is, therefore,

$$\mu = \tfrac{1}{2}.$$

Ordinarily, the perturbations are expressed in trigonometric series, according to ascending powers of the disturbing mass. But for these planets, which Gylden appropriately calls characteristic planets, the series converge very slowly, if they do not actually diverge. Such expansions are, therefore, to be avoided.

For ordinary planets we employ the method of Hansen, in which the expansions are of this type. It is easy to see that for near commensurability Hansen's integrations become indeterminate.

The differential equations of the second order contain trigonometric series of the form—

$$a \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (i\,n - i'\,n')\,t$$

where i and i' are integers, and

$$-\infty < i < +\infty \qquad o < i' \lessdot + \infty$$

The integral with respect to the time is of the form—

$$\frac{a}{(i\,n - i'\,n')^2} \begin{matrix}\sin\\ \cos\end{matrix} (i\,n - i'\,n')\,t$$

If n and n' are nearly commensurable, there will be some values of i and i' for which the denominator of the coefficient is nearly zero. For instance, for the *Hecuba* group—

$$i = 1$$
$$i' = 2,$$

gives an infinite integrating factor, for exact commensurability.

In order to solve the problem for characteristic planets, BOHLIN adapted the method of HANSEN to the case of nearly commensurable mean motions, by the use of the exponential. Evidently, if, in place of sines and cosines, we introduce the exponential through the relation—

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where

$$\theta = (i\,n - i'\,n')\,t,$$

the indeterminateness for the case

$$i\,n - i'\,n' = 0.$$

is removed.

BOHLIN retained the fundamental principles of HANSEN and his differential equations; the departure from HANSEN consists in the form of integration. The method appeared in 1896,[1] and was later extended in a French edition.[2]. In addition to the analytical developments, he published preliminary tables which are common to all groups of planets and also those which apply to the particular group $\mu = 1/3$.

H. v. ZEIPEL has taken up the *Hecuba* group according to the theory of BOHLIN, performing the integrations in essentially the same manner, but employing his own formulæ, and adding to BOHLIN's work his own determination of the constants of integration.[3]

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[1] "Formeln und Tafeln zur gruppenweisen Berechnung der allgemeinen Störungen benachbarter Planeten," *Nova Acta Reg. Soc. Sc. Upsaliensis,* Ser. III, Band XVII, 1896.

[2] "Sur le Développement des Perturbations Planétaires," *Application aux Petites Planetes,* Stockholm, 1902.

[3] Angenäherte Jupiterstörungen für die Hecuba-Gruppe," St. Petersburg, 1902.

The differential equations of HANSEN are:—

$$(1) \quad \frac{dW}{d\epsilon} = Ma_1 \frac{\delta\Omega}{d\epsilon} + Na_1\bar{r}\frac{\delta\Omega}{\delta\bar{r}}$$

$$(2) \quad \frac{dn\delta z}{d\epsilon} = (1 - e\cos\epsilon)\frac{\overline{W} + \nu^2}{1 + \overline{W}}$$

$$(3) \quad \frac{d\nu}{d\epsilon} = -\tfrac{1}{2}\,\frac{(1-\nu^2)}{1+\overline{W}}\cdot\frac{\overline{\delta W}}{\delta\Psi}$$

$$(4) \quad \sec i\,\frac{dU}{d\epsilon} = Qa^2\,\frac{\delta\Omega}{\delta z}$$

It is evident that the function W is very important and must first be obtained by integration of (1). The integration of (2), (3), (4) gives the three components of the planet's displacement due to the action of *Jupiter.*

These functions may be expressed in trigonometric series, in which the argument is of the form—

$$a\epsilon + b\theta + c\Delta + d\Sigma,$$

where ϵ is a disturbed eccentric anomaly, Δ and Σ are constants depending upon the elements π and Ω, and θ is defined through the relation—

$$\theta = \tfrac{1}{2}\,(\epsilon - e\sin\epsilon) - g',$$

to be determined from a differential equation—

$$\frac{d\theta}{d\epsilon} = f\,(W)$$

The eccentricities and mutual inclination of the planet's orbit and *Jupiter's* orbit appear explicitly as factors of the coefficients.

The series are ordered according to ascending forms of a small quantity w, defined by the relation—

$$\mu = \mu_0\,(1 - w)$$

$$\mu = \frac{n'}{n}$$

$$\mu_0 = \tfrac{1}{2}$$

Since these series are functions of μ, they may be written in terms of μ_0 and w.

$$f(\mu) = f(\mu_0) + f_1(\mu_0)\, w + f_2(\mu_0)\, w^2 + \dots$$

The possibility of developing the perturbations in such a TAYLOR'S series, upon the basis of exact commensurability, is the underlying principle of the group method.

Two important facts are to be noticed: First, that the elements appear explicitly in Δ, Σ, n, g^2, and w. In the method of HANSEN they are contained implicitly in the coefficients. Secondly, the argument θ contains implicitly the perturbations which are to be determined.

The developments proceed according to positive powers of m', and according to both positive and negative powers of w. Hence, if w is small, terms containing negative powers are numerically of lower order than is indicated by the mass factor. In place of the mass factor as the unit of order, the factor $\frac{m'}{w}$ is used. Assuming that w and $\sqrt{m'}$ are of the same order, we define successive ranks as follows:—

Rank				
0	1	$\frac{m'}{w^2}$	$\frac{m'^2}{w^4}$	. . .
1	w	$\frac{m'}{w}$	$\frac{m'^2}{w^3}$	. . .
2	w^2	m'	$\frac{m'^2}{w^{-2}}$	. . .
.	.	.	.	

In general, the mass factor is absorbed in the numerical coefficients and the tables have the following form:—

Sine Cosine	$n^p \mu^q g^{2r}$	w^s
Δ		
$2\theta + \Delta$		
$\epsilon + 2\theta + \Delta$		
$\epsilon + 4\theta + 3\Delta - \Sigma$		

Where, in general, there are three columns of coefficients, e. g. s has three values corresponding to the first three terms of the Taylor series.

In these tables the arguments are functions of both ϵ and θ. Therefore to accomplish the integration we write, for instance,

$$W = f(\theta, \epsilon)$$

$$\frac{dW}{d\epsilon} = \frac{\delta W}{\delta\epsilon} + \frac{\delta W}{\delta\theta}\,\frac{d\theta}{d\epsilon}$$

in which $\frac{dW}{d\epsilon}$ is known. The integration is then accomplished by the solution of partial differential equations in—

$$\frac{\delta W}{\delta\epsilon} \text{ and } \frac{\delta W}{\delta\theta}$$

With this purpose in view, the functions are segregated in two parts, on which always contains ϵ explicitly and one which is independent of ϵ explicitly, the latter being indicated by square brackets. For instance—

$$W = (W - [W]) + [W].$$

With these general principles, it is possible to integrate rank by rank, and the function is finally given by the sum of the integrals.

The solution of these differential equations includes the determination of constants of integration. Since the method is based upon the fundamental principles of HANSEN and the initial elements are osculating elements, these constants must be determined to satisfy the condition that the perturbations and their velocities are zero at the date of osculation. Recalling the fact that co-ordinates and their velocities for a given time determine osculating elements, it is clear, by analogy, that $n\delta z$, ν and $\frac{u}{\cos i}$ and their velocities at $t = 0$, can be absorbed by the elements.

Given osculating elements and tables for the group ½, the computer first determines this set of constant elements, which may be described as the osculating elements modified by the constants of integration. Having these elements and perturbations taken from tables, the process of computing the disturbed position is exactly the same as in HANSEN's method.

The residuals $(o - c)$, obtained in this manner for a sufficiently large number of oppositions, serve for a correction of elements.

Concerning a practical application of the method, the following general remarks are of interest:—

First, the perturbations are large. In the case of (10) *Hygiea* $n\delta z$ reaches the magnitude of 5°. Since the argument of the perturbations is a function of the disturbed position, a second approximation for $n\delta z$ is generally necessary.

Secondly, the computation, inclusive of second order in the mass, second degree (and occasionally third degree) in the eccentricities and inclinations, is quite long.

Thirdly, too much emphasis cannot be placed upon good initial elements.

Again, a successful correction of the elements is difficult. Although second order terms in the mass are included, second degree in the eccentricities and inclinations (a few of third degree), and the first three terms in the TAYLOR's expansion in w, the residuals for a large number of oppositions show an irregularity of wider range than is desirable. To take up this irregularity which may be due to some omitted terms arising from the action of *Jupiter*, or to perturbations by *Saturn*, *Mars*, or the Earth, all of which have mean motions nearly commensurable with the *Hecuba* group, it is essential that the observations used for a correction of elements shall be well distributed around the orbit.

With these general considerations in mind, the tables of v. ZEIPEL are considered applicable to planets of the *Hecuba* group, within the following limits:—

$$\phi = 10^\circ$$
$$i = 10^\circ$$
$$550'' < n < 650''$$

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